# 13 Operators

## 13.1 Preliminaries

Before we discuss operators, we have a few preliminary theorems and lemmas.

**Lemma.** *If $f \in L^2(\mathbb{S})$, i.e. periodic on $\mathbb{R}$, period $2\pi$, then $f$ is continuous-in-the-mean, i.e.*

$$\int_{\pi}^{-\pi} |f(x+t) - f(x)|^2 dx \to 0$$

*Proof.* This is a good exercise to do prove directly using integration theory. But we can do it using Fourier series.

$$f(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \qquad f(x+t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (c_k e^{ikt}) e^{ikx}$$

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So by plancherel

$$\int |f(x+t)-f(x)|^2 dx = C\sum_{n\in\mathbb{Z}} |d_k|^2 \qquad d_k = F.C. \text{ of } f(x+t)-f(x)$$
$$= C\sum_{k\in\mathbb{Z}} |1-e^{-ikt}|^2 |c_k|^2$$

For each $k$, $1-e^{ikt}$ tends to 0 as $t\to 0$. We know that the fourier series for $f(x)$ converges, so $\sum |c_k|^2 < \infty$. So given $\epsilon > 0$, $\exists N$ such that $\sum_{|k|\geq N} |c_k|^2 < \epsilon/2$, so

$$C\sum_{n\in\mathbb{Z}} |d_k|^2 \leq C\sum_{|k|<N} |1-e^{-ikt}|^2 |c_k|^2 + \sum_{|k|\geq N} |1-e^{-ikt}|^2 |c_k|^2$$

where the left tends to 0 and the right $< \epsilon$. So we are done. □

We will be discussing "compact operators" later, so we need some sort of idea of compactness. This is what the following theorem provides.

**Theorem.** *A subset $S \subset H$ of a Hilbert space is compact (here by compact we will mean sequentially compact) if and only if it is*

1. *Closed*
2. *Bounded*
3. *Satisfies the following condition*

   ***C)*** *If $\{\varphi_j\}$ is an orthonormal basis, then given $\epsilon > 0$, $\exists N$ such that*

   $$\sum_{j\geq N} |\langle \varphi_j, f\rangle|^2 < \epsilon \qquad \forall f \in S.$$

   *Note if $f$ fixed this is true, but in fact we are trying to say this for all $f$, so there is a uniformly small tail.*

*Proof.* DO THIS. □

**Definition.** A sequence $\{f_n\}$ in $H$ is weakly convergent if it is bounded and $\langle f_n, \varphi\rangle \to c$ converges in $\mathbb{C}$, $\forall \varphi \in H$

This implies: * $\langle f_n, \varphi_j\rangle \to c_j$: The Fourier coefficients converge with respect to any complete orthonormal basis

**Proposition.** 1. ** and boundedness $\Leftrightarrow$ weak convergence*

2. *Any closed bounded sequence is weakly compact in the sense that any sequence has a weakly convergence subsequence.*
3. *$f_n$ weakly convergent implies $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$, $\forall \varphi$.*

*Proof.* Of **1)**. We want to show that given $\varphi \in H$ and $\epsilon > 0$ then $\langle f_n - f_m, \varphi \rangle \to 0$.

$$\langle f_n - f_m, \varphi \rangle = \left\langle f_n - f_m, \sum_{j=1}^{N} \langle \varphi, \varphi_j \rangle \varphi_j \right\rangle + \left\langle f_n - f_m, \sum_{N+1}^{\infty} \langle \varphi, \varphi_j \rangle \varphi_j \right\rangle$$

The second half is bounded by $2 \sup_n \|f_n\| \| \sum_{N+1}^{\infty} \langle \varphi, \varphi_j \rangle \varphi_j \|$, which gets small as $N \to \infty$. And the left is bounded by $\epsilon/2$ for $n, m$ large. $\square$

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